A STOCHASTIC APPROXIMATION FOR FULLY NONLINEAR FREE BOUNDARY PROBLEMS

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Abstract. We present a stochastic numerical method for solving fully non-linear free boundary problems of parabolic type and provide a rate of convergence under reasonable conditions on the non-linearity.

1. Introduction

Option pricing problems, e.g. basket options, are generally several dimensional. In such cases, deterministic methods, e.g. finite difference, are almost intractable because the complexity increases exponentially with the dimension and one almost inevitably needs to use Monte-Carlo simulations. Moreover, most other problems in finance, e.g. pricing in incomplete markets and portfolio optimization, lead to fully non-linear PDEs. Only very recently has there been some significant development in numerically solving these non-linear PDEs, see e.g. [8], [20], [7], [17] and [12]. When the control problem also contains a stopper, e.g. in determining the super hedging price of an American option, see [13], or solving controller-and-stopper games, see [5], the non-linear PDEs have free boundaries.

For solving linear PDEs with free boundaries, i.e. in the problem of American options, Longstaff-Shwartz [16], introduced a stochastic method in which American options are approximated by Bermudan options and least squares approximation is used for doing the backward induction. The major feature in [16] is the tractability of the implementation for the scheme proposed in terms of the CPU time in high dimensional problems. The most important feature of this model that facilitates the speed is that the number of paths simulated is fixed. Simulating the paths corresponds to introducing a stochastic mesh for the space dimension and the Bermudan approximation to American options corresponds to time discretization. Stochastic mesh makes sure that the one considers the *more important* points in the state space are used in the computation of the value function, an important feature which increases the speed of convergence. So essentially, this algorithm can be thought of as an explicit finite difference scheme with stochastic mesh. One can in fact prove the convergence rate of the entire "stochastic" explicit finite difference scheme, see [9] for a survey of these results and some improvements to the original methodology of Longstaff-Shwartz.

For *semi-linear* free boundary problems a similar stochastic scheme is given through Reflected Backward Stochastic Differential Equations (RBSDE) in [17] and rate of convergence is derived to

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be $h^{1/4}$ assuming uniform ellipticity for the problem where h the mesh size of the time discretizaton. (Here the number of paths, N, that one needs to simulate increase with decreasing h and needs to be chosen in a certain way, see e.g. (3.15). This is similar to what we have in classical explicit finite difference schemes. To achieve stability, when we decrease the mesh size for time, we need to decrease the mesh size for the space variable. As we discussed above, the Monte-Carlo simulation creates a stochastic mesh.) The first result in this direction is due to [17]. Later [7] improved the result of [17] by removing the uniform ellipticity condition. Moreover, they improve the rate of convergence to $h^{1/2}$ by assuming more regularity on the obstacle function.

In this paper, we generalize the Longstaff-Schwartz methodology for numerically solving a large class of *fully non-linear* free boundary problems and show the rate of convergence of this scheme. The idea used here relies on the stochastic scheme in [12], which considers fully non-linear Cauchy problems. The proof of convergence follows the methodology of [1] with slight modifications due to the free boundary. Under an additional assumption, a rate of convergence is obtained using Krylov's method of shaking coefficients together with the switching system approximation as in [6], where a rate of approximation is obtained for classical finite difference schemes for elliptic problems with free boundaries. An appendix is provided to establish the comparison, existence and regularity results for a parabolic switching system with free boundary which is needed to provide the estimations in the rate of convergence proof. It is worth mentioning that a convergence result for classical finite difference schemes for fully nonlinear free boundary problems is provided in [19] and [18].

The rest of the paper is organized as follows: In Section 2, we present the stochastic numerical scheme. In Section 3, we present the main results, the convergence rate, and its proof. The Appendix is devoted to the analysis of non-linear switching systems with obstacles, which is an essential ingredient in the proof of our main result.

Notation. For scalars $a, b \in \mathbb{R}$, we write $a \wedge b := \min\{a, b\}$, and $a \vee b := \max\{a, b\}$. By $\mathbb{M}(n, d)$, we denote the collection of all $n \times d$ matrices with real entries. The collection of all symmetric matrices of size d is denoted $\mathbb{S}_d$, and its subset of nonnegative symmetric matrices is denoted by $\mathbb{S}_d^+$. For a matrix $A \in \mathbb{M}(n, d)$, we denote by A^{T} its transpose. For $A, B \in \mathbb{M}(n, d)$, we denote $A \cdot B := \mathrm{Tr}[A^{\mathrm{T}} B]$. In particular, for $d = 1$, A and B are vectors of $\mathbb{R}^n$ and $A \cdot B$ reduces to the Euclidean scalar product. For a suitably smooth function φ on $Q_T := (0, T] \times \mathbb{R}^d$, we define

$$|\varphi|_\infty := \sup_{(t,x)\in Q_T} |\varphi(t, x)| \quad \text{and} \quad |\varphi|_1 := |\varphi|_\infty + \sup_{Q_T\times Q_T} \frac{|\varphi(t, x) - \varphi(t', x')|}{(x - x') + |t - t'|^{\frac{1}{2}}}.$$

Finally, by $\mathbb{E}_{t,x}$ we mean the conditional expectation given $X_t = x$ for a pre-specified diffusion process X.

2. Discretization

We consider the obstacle problem

$$\min\left\{-\mathcal{L}^X v - F\left(\cdot, v, Dv, D^2 v\right), v - g\right\} = 0, \text{ on } [0, T) \times \mathbb{R}^d, \tag{2.1}$$

$$v = g, \text{ on } \{T\} \times \mathbb{R}^d, \tag{2.2}$$

where

$$\mathcal{L}^X\varphi \;\; := \;\; \frac{\partial\varphi}{\partial t} + \mu\cdot D\varphi + \frac{1}{2}a\cdot D^2\varphi,$$

and

$$F : (t, x, r, p, \gamma) \in \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \;\longmapsto\; F(x, r, p, \gamma) \in \mathbb{R},$$

is a non–linear map, μ and σ are maps from $\mathbb{R}_+ \times \mathcal{O}$ to $\mathbb{R}^d$ and $\mathbb{M}(d, d)$, respectively, $a := \sigma\sigma^{\mathrm{T}}$, $g : [0, T) \times \mathbb{R}^d \to \mathbb{R}$. We consider an $\mathbb{R}^d$-valued Brownian motion W on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = \{\mathcal{F}_t,\}_{t\in[0,T]}$ satisfies the usual conditions, and $\mathcal{F}_0$ is trivial. For a positive integer n, let $h := T/n$, $t_i = ih$, $i = 0, \ldots, n$, and consider the one step Euler discretization

$$\hat{X}_h^{t,x} := x + \mu(t, x)h + \sigma(t, x)(W_{t+h} - W_t), \tag{2.3}$$

of the diffusion X corresponding to the linear operator $\mathcal{L}^X$. Then the Euler discretization of the process X is defined by:

$$\hat{X}_{t_{i+1}} \;\; := \;\; \hat{X}_h^{t_i,\hat{X}_{t_i}}.$$

We suggest the following approximation of the value function v

$$v^h(T, x) := g(T, x) \;\text{ and }\; v^h(t_i, x) := \max\{\mathbf{T}_h[v^h](t_i, x), g(t_i, x)\} \text{ for any } x \in \mathbb{R}^d, \tag{2.4}$$

where for a given test function $\psi : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ we denote

$$\mathbf{T}_h[\psi](t, x) := \mathbb{E}_{t,x}\left[\psi(t + h, \hat{X}_{t+h})\right] + hF\left(\cdot, \mathcal{D}_h\psi\right)(t, x), \tag{2.5}$$

$$\mathcal{D}_h\psi(t_i, x) \;\; = \;\; \mathbb{E}_{t,x}\left[\psi(t + h, \hat{X}_{t+h})H_h\right], \tag{2.6}$$

where $H_h = (H_0^h, H_1^h, H_2^h)^{\mathrm{T}}$ and

$$H_0^h = 1, \quad H_1^h = \left(\sigma^{\mathrm{T}}\right)^{-1} \frac{W_h}{h}, \quad H_2^h = \left(\sigma^{\mathrm{T}}\right)^{-1} \frac{W_h W_h^{\mathrm{T}} - h\mathbf{I}_d}{h^2}\,\sigma^{-1}.$$

Remark 2.1. The reasoning behind (2.6) can be found in Lemma 2.1 in [12].

3. Asymptotics of the discrete-time approximation

In this section, we present the convergence and the rate of convergence result for the scheme introduced in (2.4), and the assumptions needed for these results.

3.1. **The main results.** The proof of the convergence follow the general methodology of Barles and Souganidis [1], and requires that the nonlinear PDE (2.1) satisfies the comparison principle in viscosity sense.

We recall that an upper-semicontinuous (resp. lower-semicontinuous) function $\underline{v}$ (resp. $\overline{v}$) on $[0, T]\times\mathbb{R}^d$, is called a viscosity subsolution (resp. supersolution) of (2.1) if for any $(t, x) \in [0, T]\times\mathbb{R}^d$ and any smooth function φ satisfying

$$0 = (\underline{v} - \varphi)(t, x) = \max_{[0,T]\times\mathbb{R}^d}(\underline{v} - \varphi)\left(\text{resp. } 0 = (\overline{v} - \varphi)(t, x) = \min_{[0,T]\times\mathbb{R}^d}(\overline{v} - \psi)\right),$$

we have:

- if $t < T$ and $x \in \mathbb{R}^d$

$$\min\left\{-\mathcal{L}^X\varphi - F(t,x,\mathcal{D}\varphi(t,x)), \varphi - g\right\} \quad \leq \quad (\text{resp. } \geq) \quad 0,$$

- if $t = T$, $\underline{v} - g \leq 0$ (resp. $\overline{v} - g \geq 0$).

Remark 3.1. Note that the above definition is not symmetric for sub and supersolutions. More precisely, for a subsolution we need to have either

$$-\mathcal{L}^X\varphi - F(t,x,\mathcal{D}\varphi(t,x)) \leq 0 \text{ or } \varphi - g \leq 0.$$

However, for a supersolutions we need to have both

$$-\mathcal{L}^X\varphi - F(t,x,\mathcal{D}\varphi(t,x)) \geq 0 \text{ and } \varphi - g \geq 0.$$

Definition 3.2. *We say that* (2.1) *has comparison for bounded functions if for any bounded upper semicontinuous subsolution* $\underline{v}$ *and any bounded lower semicontinuous supersolution* $\overline{v}$ *on* $[0,T) \times \mathbb{R}^d$, *satisfying* $\underline{v}(T,\cdot) \leq \overline{v}(T,\cdot)$, *we have* $\underline{v} \leq \overline{v}$.

We denote by F_r, F_p and F_γ the partial gradients of F with respect to r, p and γ, respectively. We also denote by F_γ^- the pseudo-inverse of the non-negative symmetric matrix F_γ.

Assumption F (i) *The nonlinearity* F *is Lipschitz-continuous with respect to* (x,r,p,γ) *uniformly in* t, *and* $|F(\cdot,\cdot,0,0,0)|_\infty < K$ *for some positive constant* K;

(ii) F is elliptic and dominated by the diffusion of the linear operator $\mathcal{L}^X$, i.e.

$$\nabla_\gamma F \leq a \quad \text{on} \quad \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d; \tag{3.1}$$

(iii) $F_p \in \mathrm{Image}(F_\gamma)$ and $\left|F_p^{\mathrm{T}} F_\gamma^- F_p\right|_\infty < K$;

(iv) $F_r - \frac{1}{4} F_p^T F_\gamma^- F_p \geq 0$.

Remark 3.3. Assumption $\mathbf{F}$(iv) is made for the sake of simplicity of the presentation. It implies the monotonicity of the above scheme. If this assumption is not made, one can carry out the analysis in [12, Remark 3.13, Theorem 3.12, and Lemma 3.19] and approximate the solution of the non-monotone scheme with the solution of an appropriate monotone scheme.

Theorem 3.4 (Convergence)**.** *Suppose that Assumption* $\mathbf{F}$ *holds, that* $|\mu|_1 + |\sigma|_1 < \infty$, *and that* σ *is invertible. Also, assume that the fully nonlinear PDE* (2.1) *has comparison for bounded functions. Then for every bounded function* g *Lipschitz on* x *and* $\frac{1}{2}$*−Hölder on* t, *there exists a bounded function* v *such that* $v^h \longrightarrow v$ *locally uniformly. Moreover,* v *is the unique bounded viscosity solution of problem* (2.1)-(2.2).

By imposing the following stronger assumption, we are able to derive a rate of convergence for the fully non–linear PDE.

Assumption HJB *The nonlinearity F satisfies Assumption* **F***(ii)-(iii), and is of the Hamilton-Jacobi-Bellman type:*

$$\begin{aligned}\frac{1}{2}a\cdot\gamma+b\cdot p+F(t,x,r,p,\gamma) &= \inf_{\alpha\in\mathcal{A}}\{\mathcal{L}^\alpha(t,x,r,p,\gamma)\},\\ \mathcal{L}^\alpha(t,x,r,p,\gamma) &:= \frac{1}{2}Tr[\sigma^\alpha\sigma^{\alpha\mathrm{T}}(t,x)\gamma]+b^\alpha(t,x)p+c^\alpha(t,x)r+f^\alpha(t,x),\end{aligned}$$

where the functions μ, σ, σ^α, b^α, c^α and f^α satisfy:

$$|\mu|_\infty+|\sigma|_\infty+\sup_{\alpha\in\mathcal{A}}\left(|\sigma^\alpha|_1+|b^\alpha|_1+|c^\alpha|_1+|f^\alpha|_1\right) \quad < \quad \infty.$$

Assumption HJB+ *The nonlinearity F satisfies* **HJB***, and for any $\delta>0$, there exists a finite set $\{\alpha_i\}_{i=1}^{M_\delta}$ such that for any $\alpha\in\mathcal{A}$*

$$\inf_{1\le i\le M_\delta}|\sigma^\alpha-\sigma^{\alpha_i}|_\infty+|b^\alpha-b^{\alpha_i}|_\infty+|c^\alpha-c^{\alpha_i}|_\infty+|f^\alpha-f^{\alpha_i}|_\infty \quad \le \quad \delta.$$

Remark 3.5. Assumption HJB+ is satisfied if $\mathcal{A}$ is a compact separable topological space and $\sigma^\alpha(\cdot)$, $b^\alpha(\cdot)$, $c^\alpha(\cdot)$ and $f^\alpha(\cdot)$ are continuous maps from $\mathcal{A}$ to $C_b^{\frac{1}{2},1}$, the space of bounded maps which are Lipschitz in x and $\frac{1}{2}$–Hölder in t.

Theorem 3.6 (Rate of Convergence)**.** *Assume that the final condition g is bounded Lipschitz on x and $\frac{1}{2}-$Hölder on t. Then, there is a constant $C>0$ such that:*

(i) *under Assumption* HJB*, we have $v-v^h\le Ch^{1/4}$,*
(ii) *under the stronger condition* HJB+*, we also have $-Ch^{1/10}\le v-v^h$.*

It is worth mentioning that in the finite difference literature, the rate of convergence is usually stated in terms of the discretization in the space variable, i.e. $|\Delta x|$, and the time step, i.e. $|\Delta t|$ equals $|\Delta x|^2$. In our context, the stochastic numerical scheme (2.4) is only discretized in time with time step h. Therefore, the rates of convergence in Theorem 3.6 corresponds to the rates $|\Delta x|^{1/2}$ and $|\Delta x|^{1/5}$, respectively.

3.2. **Proof of the convergence result.** The proof Theorem 3.4, similar to the proof of Theorem 3.6 in [12], is based on the result of [1] which requires the scheme to be consistent, monotone and stable. Notice that from Lemmas 3.11, 3.12, and 3.14 in [12], we already know the consistency, monotonicity and stability for the scheme without obstacle, i.e., $\mathbf{T}_h$ satisfies

- Let φ be a smooth function with bounded derivatives. Then for all $(t,x)\in[0,T]\times\mathbb{R}^d$:

$$\lim_{\substack{(t',x')\to(t,x)\\(h,c)\to(0,0)\\t'+h\le T}}\frac{[c+\varphi](t',x')-\mathbf{T}_h[c+\varphi](t',x')}{h} \quad = \quad -\left(\mathcal{L}^X\varphi+F(\cdot,\varphi,D\varphi,D^2\varphi)\right)(t,x).$$

- Let $\varphi,\psi:\ [0,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}$ be two bounded functions. Then:

$$\varphi\le\psi \quad\Longrightarrow\quad \mathbf{T}_h[\varphi](t,x)\le\mathbf{T}_h[\psi](t,x).$$

- If g is bounded, the family $(\overline{v}^h)_h$ defined by
$$\overline{v}^h(T,x) = g(T,x) \quad\text{and}\quad \overline{v}^h(t_i,x) = \mathbf{T}_h[\overline{v}^h](t_i,x)$$
is bounded, uniformly in h.

In the next result, we will show that $(v^h)_h$ is also bounded, uniformly in h, i.e., this sequence of functions is stable in the sense of [1].

Lemma 3.7. *The family $(v^h)_h$ defined by (2.4) is bounded, uniformly in h.*

Proof. Due to the monotonicity of $\mathbf{T}_h$, we already have that $\overline{v}^h \le v^h$. Let $C_i = |v^h(t_i,\cdot)|_\infty$. By the argument in the proof of Lemma 3.14 in [12], $|T_h[v^h](t_i,\cdot)|_\infty \le C_{i+1}(1+Ch)+Ch$. Therefore,
$$C_i \le \max\{|g|_\infty, C_{i+1}(1+Ch)+Ch\} \le \max\{C_{i+1}, |g|_\infty\}(1+Ch)+Ch.$$
Using a backward induction one could obtain that $C_i \le Ce^{CT}$ for some constant C independent of h. $\square$

Let us define
$$\begin{aligned} v_*(t,x) &:= \liminf_{\delta\to 0}\liminf_{h\to 0}\inf\{v^h(t,y) : |x-y|+|s-t|\le\delta, s\in\{0,h,\cdots\}\cap[0,T]\}, \quad\text{and}\\ v^*(t,x) &:= \limsup_{\delta\to 0}\limsup_{h\to 0}\sup\{v^h(t,y) : |x-y|+|s-t|\le\delta, s\in\{0,h,\cdots\}\cap[0,T]\}. \end{aligned} \tag{3.2}$$
We are going to prove that v^* and v_* are respectively sub and supersolutions of (2.1)–(2.2). We first provide the argument for v^* at an interior point $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$. We will present the subsolution property at $t_0=T$ separately.

Assume that $v^*(t_0,x_0) > g(t_0,x_0)$, otherwise $v^*(t_0,x_0) = g(t_0,x_0)$ and the subsolution property is readily satisfied. Let ϕ be a smooth function such that
$$0 = \max_{[0,T]\times\mathbb{R}^d}(v^*-\phi) = (v^*-\phi)(t_0,x_0),$$
and that at (t_0,x_0) is global strict maximum is attained. (Here, we can assume that the global maximum is attained at our given point without loss of generality, thanks to Lemma 3.7.) Let (t_n,x_n) be the global maximum of $v^{h_n}-\phi$. The fact that (t_0,x_0) is a global maximum of $v^*-\phi$ imply that there exists a subsequence, which we still denote by (t_n,x_n) such that $(t_n,x_n)\to(t_0,x_0)$ and $v^{h_n}(t_n,x_n)\to v^*(t_0,x_0)$.

Let $\xi_n := \max(v^{h_n}-\phi)$. Since $v^{h_n}\le\phi+\xi_n$, the monotonicity of $\mathbf{T}_{h_n}$ implies that $\mathbf{T}_{h_n}[v^{h_n}](t_n,x_n)\le\mathbf{T}_{h_n}[\phi+\xi_n](t_n,x_n)$. Observe that by $v^*(t_0,x_0) > g(t_0,x_0)$, one can deduce that for large n, $v^{h_n}(t_n,x_n) > g(t_n,x_n)$ holds true. Therefore,
$$v^{h_n}(t_n,x_n) = \mathbf{T}_{h_n}[v^{h_n}](t_n,x_n) \le \mathbf{T}_{h_n}[\phi+\xi_n](t_n,x_n).$$
Because $v^{h_n}(t_n,x_n) = \phi(t_n,x_n)+\xi_n$,
$$\phi(t_n,x_n)+\xi_n-\mathbf{T}_{h_n}[\phi+\xi_n](t_n,x_n)\le 0.$$
Dividing by h_n and taking the limit as $n\to\infty$, the consistency of the scheme implies that
$$-\left(\mathcal{L}^X\varphi + F(\cdot,\varphi,D\varphi,D^2\varphi)\right)(t_0,x_0)\le 0.$$

To prove that v_* is a supersolution, we need to show that that $v^h(t,x) \geq g(t,x)$ and $-\mathcal{L}^X\varphi - F(t,x,\mathcal{D}\varphi(t,x)) \geq 0$ holds in the viscosity sense. Let ϕ be a smooth function such that

$$0 = \min_{[0,T]\times\mathbb{R}^d}(v_* - \phi) = (v_* - \phi)(t_0,x_0),$$

and the (t_0,x_0) is a global strict minimum point (again this is as above without loss of generality). Therefore, there exists a sequence $\{(t_n,x_n)\}$, such that $(t_n,x_n) \to (t_0,x_0)$, $v^{h_n}(t_n,x_n) \to v^*(t_0,x_0)$, $\xi_n := \min(v^{h_n} - \phi) = (v^{h_n} - \phi)(t_n,x_n) \to 0$, and (t_n,x_n) is a global minimum of $v^{h_n} - \phi$.
Therefore, $v^{h_n} \geq \phi + \xi_n$. By monotonicity of the scheme, $\mathbf{T}_{h_n}[v^{h_n}](t_n,x_n) \geq \mathbf{T}_{h_n}[\phi+\xi_n](t_n,x_n)$. Therefore, by the definition of v^h in (2.4),

$$v^{h_n}(t_n,x_n) \geq \mathbf{T}_{h_n}[v^{h_n}](t_n,x_n) \geq \mathbf{T}_{h_n}[\phi+\xi_n](t_n,x_n).$$

Because $v^{h_n}(t_n,x_n) = \phi(t_n,x_n) + \xi_n$,

$$\phi(t_n,x_n) + \xi_n - \mathbf{T}_{h_n}[\phi+\xi_n](t_n,x_n) \geq 0.$$

Dividing by h_n and taking the limit as $n \to \infty$, we obtain

$$-\left(\mathcal{L}^X\varphi + F(\cdot,\varphi,D\varphi,D^2\varphi)\right)(t_0,x_0) \geq 0.$$

If $t_0 = T$, observe that by monotonicity of $\mathbf{T}$, $\bar{v}^h \leq v^h$. Therefore, by Lemma 3.17 in [12],

$$g(T,x_0) = \liminf_{(h,t',x')\to(0,T,x_0)} \bar{v}^h(t',x') \leq v_*(T,x_0),$$

which completes the proof of the supersolution argument.
Finally, the following Lemma shows that $v^*(T,\cdot) = g(T,\cdot)$ which completes the subsolution argument.

Lemma 3.8. $|v^h(t_i,x) - g(T,x)| \leq C(T-t_i)^{\frac{1}{2}}$.

Proof. By replacing the stopping time in the Lemma 3.11 [11] by a new stopping time defined by $\hat{\tau} := \min\{t_j | v^h(t_j,\hat{X}_{t_j}) = g(t_j,\hat{X}_{t_j})\}$, it follows that there is a positive constant C such that

$$|v^h(t_i,x) - g(t_i,x)| \leq C\sqrt{T-t_i}.$$

Then, the result follows from $\frac{1}{2}$-Hölder continuity of g with respect to t. $\square$

3.3. **Derivation of the rate of convergence.** The proof of Theorem 3.6 is based on Barles and Jakobsen [3], which uses switching systems approximation and the Krylov method of shaking coefficients [14]. This has been adapted to classical finite difference schemes for elliptic obstacle problems (free boundary) problems in [6]. In order to use the method, we need to introduce a comparison principle for the scheme which we will undertake next.

Proposition 3.9. *Let Assumption* $\mathbf{F}$ *hold and set* $\beta := |F_r|_\infty$. *Consider two arbitrary bounded functions* φ *and* ψ *satisfying:*

$$\min\left\{h^{-1}\left(\varphi - \mathbf{T}_h[\varphi]\right), \varphi - g\right\} \leq g_1 \quad \text{and} \quad \min\left\{h^{-1}\left(\psi - \mathbf{T}_h[\psi]\right), \psi - g\right\} \geq g_2 \tag{3.3}$$

for some bounded functions g_1 and g_2. Then, for every $i = 0, \cdots, n$:

$$(\varphi - \psi)(t_i, x) \quad \leq \quad e^{\beta(T-t_i)} \left(|(\varphi - \psi)^+(T, \cdot)|_\infty + (1 + (T - t_i))|(g_1 - g_2)^+|_\infty \right). \tag{3.4}$$

Proof. We may assume without loss of generality that

$$\varphi(T, \cdot) \leq \psi(T, \cdot) \quad \text{and} \quad g_1 \leq g_2.$$

Otherwise, one may take $\bar{\psi}(t, x) = \psi(t, x) + \delta(t)$, where

$$\delta(t) = e^{\beta(T-t)} \left(|(\varphi - \psi)^+(T, \cdot)|_\infty + (1 + (T - t))|(g_1 - g_2)^+|_\infty \right).$$

Then, by Lemma 3.21 in [12],

$$h^{-1} \left(\bar{\psi} - \mathbf{T}_h[\bar{\psi}] \right) \geq h^{-1} \left(\psi - \mathbf{T}_h[\psi] \right) + |(g_1 - g_2)^+|_\infty \geq g_1 \vee g_2.$$

On the other hand, by the definition of $\bar{\psi}$ and δ, $\bar{\psi} - g \geq g_2 + |(g_1 - g_2)^+|_\infty \geq g_1 \vee g_2$. Therefore,

$$\min \left\{ h^{-1} \left(\bar{\psi} - \mathbf{T}_h[\bar{\psi}] \right), \bar{\psi} - g \right\} \;\geq\; g_1 \vee g_2.$$

Now, we proceed by induction. We assume that for some i, $\varphi(t_i + h, \cdot) \leq \psi(t_i + h, \cdot)$.
Case 1. If,

$$\min \left\{ h^{-1} \left(\varphi - \mathbf{T}_h[\varphi] \right), \varphi - g \right\} (t_i, x) \quad = \quad h^{-1} \left(\varphi - \mathbf{T}_h[\varphi] \right) (t_i, x),$$

then, since

$$\min \left\{ h^{-1} \left(\psi - \mathbf{T}_h[\psi] \right), \psi - g \right\} (t_i, x) \quad \leq \quad h^{-1} \left(\psi - \mathbf{T}_h[\psi] \right) (t_i, x),$$

we have $h^{-1} \left(\varphi - \mathbf{T}_h[\varphi] \right) (t_i, x) \leq h^{-1} \left(\psi - \mathbf{T}_h[\psi] \right) (t_i, x)$. Due to monotonicity of $\mathbf{T}_h$, we can deduce that $\varphi(t_i, x) \leq \psi(t_i, x)$.
Case 2. If, on the other hand,

$$\min \left\{ h^{-1} \left(\varphi - \mathbf{T}_h[\varphi] \right), \varphi - g \right\} (t_i, x) \quad = \quad (\varphi - g)(t_i, x),$$

then, we have

$$\min \left\{ h^{-1} \left(\psi - \mathbf{T}_h[\psi] \right), \psi - g \right\} (t_i, x) \quad \leq \quad (\psi - g)(t_i, x),$$

we have $\varphi(t_i, x) \leq \psi(t_i, x)$. □

3.3.1. *Proof of Theorem 3.6 (i).* Under Assumption HJB, we can build a bounded subsolution v^ε of the nonlinear PDE, by the method of shaking the coefficients, see [3], [6], [15], and the references therein. More precisely, consider the following equation

$$\min \left\{ -\mathcal{L}^X v - \inf_{0 < s < \varepsilon^2, |y| < \varepsilon} F\left(t - s, x + y, v, Dv, D^2 v\right), v - g \right\} \;= 0, \text{ on } [0, T) \times \mathbb{R}^d, \tag{3.5}$$

$$v \;= g, \text{ on } \{T\} \times \mathbb{R}^d. \tag{3.6}$$

By Theorem A.5, there exists a unique bounded solution v^ε, in the class of function with at most linear growth, to (3.5)-(3.6). This solution, due to Theorem A.6 is also Lipschitz in x, $1/2-$Hölder continuous in t. It is not hard to see that v^ε is a subsolution to (2.1)-(2.2) and by Theorem A.3 approximates v uniformly, i.e., there exists a positive constant C such that $v - C\varepsilon \leq v^\varepsilon \leq v$.

Let $\rho(t,x)$ be a C^∞ positive function supported in $\{(t,x) : t \in [0,1], |x| \le 1\}$ with unit mass, and define

$$w^\varepsilon(t,x) := v^\varepsilon * \rho^\varepsilon \quad \text{where} \quad \rho^\varepsilon(t,x) := \frac{1}{\varepsilon^{d+2}}\rho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right). \tag{3.7}$$

It follows that $|w^\varepsilon - v| \le C\varepsilon$. From the convexity of the operator $-F$ and the fact that $w^\varepsilon \in C^\infty$, w^ε is a classical subsolution of (2.1) on $U := \{\ (t,x) \mid g(t-s, x+y) < v^\varepsilon(t-s, x+y);\ \text{for any } s \in [0,\varepsilon^2) \text{ and } |y| < \varepsilon\}$, see e.g. the Appendix in [4]. Moreover, since v^ε is Lipschitz in x and $1/2-$Hölder continuous in t, it follows that

$$\left|\partial_t^{\beta_0} D^\beta w^\varepsilon\right| \le C\varepsilon^{1-2\beta_0-|\beta|_1} \quad \text{for any} \quad (\beta_0, \beta) \in \mathbb{N} \times \mathbb{N}^d \setminus \{0\}, \tag{3.8}$$

where $|\beta|_1 := \sum_{i=1}^d \beta_i$, and $C > 0$ is some constant. As a consequence of the consistency of $\mathbf{T}_h$, see Lemma 3.11 in [12], we know that

$$\mathcal{R}_h[w^\varepsilon](t,x) \ := \ \frac{w^\varepsilon(t,x) - \mathbf{T}_h[w^\varepsilon](t,x)}{h} \ + \ \mathcal{L}^X w^\varepsilon(t,x) + F(\cdot, w^\varepsilon, Dw^\varepsilon, D^2 w^\varepsilon)(t,x) \le Ch\varepsilon^{-3}.$$

From this estimate together with the subsolution property of w^ε, we see that $w^\varepsilon \le \mathbf{T}_h[w^\varepsilon] + Ch^2\varepsilon^{-3}$ holds true on U. In addition, by the regularity properties of g, one can see that $w^\varepsilon \le g + C\varepsilon$ on $[0,T] \times \mathbb{R}^d \setminus U$. Therefore,

$$\min\left\{\frac{w^\varepsilon(t,x) - \mathbf{T}_h[w^\varepsilon](t,x)}{h}, w^\varepsilon - g\right\} \le C_1(\varepsilon + \varepsilon^{-3}h).$$

Then, it follows from Proposition 3.9 that

$$w^\varepsilon - v^h \le C|(w^\varepsilon - v^h)(T,.)|_\infty + C_1(\varepsilon + h\varepsilon^{-3}) \le C(\varepsilon + h\varepsilon^{-3}). \tag{3.9}$$

Therefore, $v - v^h \le v - w^\varepsilon + w^\varepsilon - v^h \le C(\varepsilon + h\varepsilon^{-3})$. Minimizing the right hand-side estimate over $\varepsilon > 0$, we obtain $v - v^h \le Ch^{1/4}$. □

3.3.2. *Proof of Theorem 3.6 (ii).* To prove the lower bound on the rate of convergence, we will use Assumption **HJB+** and build a switching system approximation to the solution of the nonlinear obstacle problem (2.1)-(2.2). This proof method has been used for Cauchy problems in [3] and [12]. For obstacle problems, this method is used in the elliptic case by [6] for the classical finite difference schemes. We apply this methodology for parabolic obstacle problems to prove the lower bound for the convergence rate of our stochastic finite difference scheme. We split the proof into the following steps:

(1) Approximating the solution to (2.1)-(2.2) by a switching system, which relies on Theorem A.3, the continuous dependence result for switching systems with obstacle.
(2) Building an almost everywhere smooth supersolution to (2.1)-(2.2) using the mollification of the solution to the switching system.
(3) Using Proposition 3.9, the comparison principle for the scheme, to bound the difference of the supersolution obtained in Step 2 and the approximate solution obtained from the scheme.

Step 1. Consider the following switching system:

$$\min\left\{\max\left\{-v_t^{(i)}\sup_{0<s<\varepsilon^2,|y|<\varepsilon}\mathcal{L}^{\alpha_i}v^{\varepsilon,i}(\cdot-s,\cdot+y),v^{\varepsilon,i}-\mathcal{M}^{(i)}v^{\varepsilon}\right\},v^{\varepsilon,i}-g\right\}(t,x)=0, \tag{3.10}$$

$$v^{\varepsilon,i}(T,\cdot)=g(T,\cdot), \tag{3.11}$$

where $v^{\varepsilon}=(v^{\varepsilon,i})_{i=1}^{M}$, $\mathcal{M}^{(i)}v^{\varepsilon}=\min_{j:j\neq i}\{v^{\varepsilon,j}+k\}$, k is a non-negative constant, α_i's, for $i=1,\cdots,M$, are as in assumption **HJB+**, and $\mathcal{L}^{\alpha_i}\varphi:=\frac{1}{2}\mathrm{Tr}\left[a^{\alpha_i}(t,x)D^2\varphi\right]+b^{\alpha_i}(t,x)D\varphi+c^{\alpha_i}(t,x)\varphi+f^{\alpha_i}(t,x)$.

By Theorem A.5 the viscosity solution v^{ε} to (3.10)-(3.11) exists and by Theorem A.6 is Lipschitz continuous on x and $\frac{1}{2}$-Hölder continuous on t. Moreover, by using Assumption **HJB+**, Theorem A.3 and Remark A.1, one can approximate the solution to (2.1)-(2.2) by the solution to (3.10)-(3.11), see Theorem 3.4 [6] and the proof of Theorem 2.3 in [3] for more details. More precisely, there exists a positive constant C such that

$$|v-v^{\varepsilon,i}|_{\infty}\leq C(\varepsilon+k^{\frac{1}{3}}).$$

Step 2. Let $v_{\varepsilon}^{(i)}:=v^{\varepsilon,i}*\rho^{\varepsilon}$, where $\{\rho^{\varepsilon}\}$ is as in (3.7). As in Lemma 4.2 in [6] and Lemma 3.4 in [3] for $\varepsilon\leq\left(12\sup_i|v_{\varepsilon}^{(i)}|_1\right)^{-1}k$, the almost everywhere smooth function $w_{\varepsilon}:=\min\limits_i v_{\varepsilon}^{(i)}$ is a supersolution to

$$-\mathcal{L}^{X}v-F\left(\cdot,v,Dv,D^2v\right)\geq 0. \tag{3.12}$$

Moreover, for any $(t,x)\in[0,T)\times\mathbb{R}^d$, if $i_0\in\operatorname*{argmin}\limits_i v_{\varepsilon}^{(i)}(t,x)$, we have $v_{\varepsilon}^{(i_0)}(t,x)<v_{\varepsilon}^{(i)}(t,x)+k$. Therefore, for all i we have

$$(w_{\varepsilon}-v)(t,x)=(v_{\varepsilon}^{(i_0)}-v)(t,x)\leq(v_{\varepsilon}^{(i_0)}-v_{\varepsilon}^{(i)})(t,x)+(v_{\varepsilon}^{(i)}-v)(t,x)\leq k+C(\varepsilon+k^{\frac{1}{3}}).$$

Choosing $k=C_1\varepsilon$ with $C_1=12\sup_i|v_{\varepsilon}^{(i)}|_1$, one can write

$$(w_{\varepsilon}-v)(t,x)\leq C\varepsilon^{\frac{1}{3}}. \tag{3.13}$$

Step 3. Notice that w_{ε} is almost everywhere smooth and therefore, the subsolution property holds true almost everywhere in classic sense. Moreover, since (3.8) is almost everywhere satisfied by w_{ε}, one can conclude that

$$\mathcal{R}_h[w_{\varepsilon}](t,x)\;:=\;\frac{w_{\varepsilon}(t,x)-\mathbf{T}_h[w_{\varepsilon}](t,x)}{h}\;+\;\mathcal{L}^{X}w_{\varepsilon}(t,x)+F(\cdot,w_{\varepsilon},Dw_{\varepsilon},D^2w_{\varepsilon})(t,x)\geq -Ch\varepsilon^{-3},$$

therefore due to (3.12), $\frac{w_{\varepsilon}(t,x)-\mathbf{T}_h[w_{\varepsilon}](t,x)}{h}\geq -Ch\varepsilon^{-3}$ holds true almost everywhere. By Proposition 3.9, one can get

$$(v^h-w^{\varepsilon})(t,x)\leq Ch\varepsilon^{-3}. \tag{3.14}$$

Now, (3.13) and (3.14) yields

$$(v^h-v)(t,x)\leq C(\varepsilon^{\frac{1}{3}}+\varepsilon^{-3}h).$$

By minimizing on $\varepsilon>0$, the desired lower bound is obtained. □

Remark 3.10 (Stochastic scheme)**.** Scheme (2.4) produces a deterministic approximate solution. However, in practice, we approximate the expectations in (2.5) based on a randomly generated set of sample paths of the process $\hat{X}$. As a result, the approximate solution is not deterministic anymore. By following the line of argument in Section 4 in [12], one can show the almost sure convergence of this stochastic approximate solution and even provide the same rate of convergence in $\mathbb{L}^p(\Omega, \mathbb{P})$.

More precisely, assume that $\mathbb{E}$ is approximated by $\hat{\mathbb{E}}^N$ where N denotes the number of sample paths. Suppose that for some $p \geq 1$, there exist constants $C_b, \lambda, \nu > 0$ such that $\left\| \hat{\mathbb{E}}^N[R] - \mathbb{E}[R] \right\|_p \leq C_b h^{-\lambda} N^{-\nu}$ for a suitable class of random variables R bounded by b. By replacing $\mathbb{E}$ with $\hat{\mathbb{E}}^N$ in the scheme (2.4), one obtains a stochastic approximate solution $\hat{v}^h_N$. Then, if we choose $N = N_h$ which is chosen to satisfy $\lim_{h\to 0} N_h^\nu h^{\lambda+2} = \infty$, then under assumptions of Theorem 3.4

$$\hat{v}^h_{N_h}(\cdot, \omega) \longrightarrow v \qquad \text{locally uniformly,}$$

for almost every ω where v is the unique viscosity solution of (2.1)-(2.2). In addition, if

$$\lim_{h\to 0} N_h^\nu h^{\lambda + \frac{21}{10}} > 0, \tag{3.15}$$

we have that $\|v - \hat{v}^h_{N_h}\|_p \leq C h^{1/10}$, under the assumptions of Theorem 3.6.

A. Appendix: A switching system with an obstacle

In this section, we will provide some results needed in the Section 3.3. In particular, we present a continuous dependence result for the switching system with obstacle and as a corollary a comparison result, which provides the uniqueness of the solution. Then, the existence and regularity of the solutions to the switching systems are provided.

Consider the following system of PDEs for $v = (v^{(i)})_{i=1}^M$:

$$\begin{aligned} \min\left\{\max\left\{-v^{(i)}_t - F_i(\cdot, v^{(i)}, Dv^{(i)}, D^2 v^{(i)}), v^{(i)} - \mathcal{M}^{(i)} v\right\}, v^{(i)} - g\right\} = 0, \quad \text{for } i = 1, \cdots, M;\\ v^{(i)}(T, \cdot) = g(T, \cdot). \end{aligned} \tag{A.1}$$

We also need to consider a variant of equation (A.1) as follows:

$$\begin{aligned} \min\left\{\max\left\{-v^{(i)}_t - \hat{F}_i(\cdot, v^{(i)}, Dv^{(i)}, D^2 v^{(i)}), v^{(i)} - \mathcal{M}^{(i)} v\right\}, v^{(i)} - \hat{g}\right\} = 0, \quad \text{for } i = 1, \cdots, M;\\ v^{(i)}(T, \cdot) = \hat{g}(T, \cdot). \end{aligned} \tag{A.2}$$

Assumption HJB-S. *We assume that in* (A.1) *and* (A.2)

$$F_i(\cdot, v^{(i)}, Dv^{(i)}, D^2 v^{(i)}) = \inf_{\alpha \in \mathcal{A}^i} \mathcal{L}^{i,\alpha} v^{(i)} \quad \textit{and} \quad \hat{F}_i(\cdot, v^{(i)}, Dv^{(i)}, D^2 v^{(i)}) = \inf_{\alpha \in \mathcal{A}^i} \hat{\mathcal{L}}^{i,\alpha} v^{(i)}, \tag{A.3}$$

$\mathcal{M}^{(i)} v = \min_{j: j \neq i}\{v_j + k\}$*, k is a non-negative constant, and*

$$\begin{aligned} \mathcal{L}^{i,\alpha}\varphi(x) &:= \frac{1}{2}\mathrm{Tr}\left[a^\alpha_i(t,x) D^2\varphi\right] + b^\alpha_i(t,x) D\varphi + c^\alpha_i(t,x)\varphi + f^\alpha_i(t,x),\\ \hat{\mathcal{L}}^{i,\alpha}\varphi(x) &:= \frac{1}{2}\mathrm{Tr}\left[\hat{a}^\alpha_i(t,x) D^2\varphi\right] + \hat{b}^\alpha_i(t,x) D\varphi + \hat{c}^\alpha_i(t,x)\varphi + \hat{f}^\alpha_i(t,x). \end{aligned}$$

Moreover,

$$L := |g|_1 + |\hat{g}|_1 + \sup_{\alpha \in \bigcup_i \mathcal{A}^i} \Big(|\sigma^\alpha|_1 + |b^\alpha|_1 + |c^\alpha|_1 + |f^\alpha|_1 + |\hat{\sigma}^\alpha|_1 + |\hat{b}^\alpha|_1 + |\hat{c}^\alpha|_1 + |\hat{f}^\alpha|_1 \Big) < \infty,$$

and for all $\alpha \in \bigcup_i \mathcal{A}^i$*, we have* $c_i^\alpha \geq 0$ *and* $\hat{c}_i^\alpha \geq 0$.

Remark A.1. *For the sake of simplicity in Assumption* **HJB-S***, we only included the nonlinearities of infinum type. However, all the results of this appendix still hold if we assume that*

$$F_i(\cdot, v^{(i)}, Dv^{(i)}, D^2 v^{(i)}) = \inf_{\alpha \in \mathcal{A}^i} \sup_{\beta \in \mathcal{B}^i} \mathcal{L}^{i,\alpha,\beta} \quad \textit{and} \quad \hat{F}_i(\cdot, v^{(i)}, Dv^{(i)}, D^2 v^{(i)}) = \inf_{\alpha \in \mathcal{A}^i} \sup_{\beta \in \mathcal{B}^i} \hat{\mathcal{L}}^{i,\alpha,\beta}, \quad \text{(A.4)}$$

$$|g|_1 + |\hat{g}|_1 + \sup_{\substack{\alpha \in \bigcup_i \mathcal{A}^i \\ \beta \in \bigcup_i \mathcal{B}^i}} \Big(|\sigma_i^{\alpha,\beta}|_1 + |b_i^{\alpha,\beta}|_1 + |c_i^{\alpha,\beta}|_1 + |f_i^{\alpha,\beta}|_1 + |\hat{\sigma}_i^{\alpha,\beta}|_1 + |\hat{b}_i^{\alpha,\beta}|_1 + |\hat{c}_i^{\alpha,\beta}|_1 + |\hat{f}_i^{\alpha,\beta}|_1 \Big) < \infty,$$

and for all $\alpha \in \bigcup_i \mathcal{A}^i$ *and* $\beta \in \bigcup_i \mathcal{B}^i$*, we have* $c_i^{\alpha,\beta} \geq 0$ *and* $\hat{c}_i^{\alpha,\beta} \geq 0$*. This remark is also valid if we change the order of* inf *and* sup *in* (A.4)*.*

Lemma A.2. *Let* $u = (u^{(i)})_i$ *and* $v = (v^{(i)})_i$ *be respectively the upper semicontinuous subsolution and the lower semicontinuous supersolution of* (A.1) *and* (A.2)*, and assume that* $\varphi(t, x, y)$ *is a smooth function bounded from below. Define*

$$\begin{aligned}
\psi^{(i)}(t, x, y) &\equiv u^{(i)}(t, x) - v^{(i)}(t, y) - \varphi(t, x, y), \\
\mathcal{J}_1 &:= \left\{ j \,\middle|\, \exists (t', x', y') : \sup_{i,t,x,y} \psi^{(i)}(t, x, y) = \psi^{(j)}(t', x', y') \right\}, \\
\mathcal{J}_2(t, x) &:= \left\{ j \,\middle|\, u^{(j)}(t, x) \leq g(t, x) \right\}.
\end{aligned}$$

Suppose that there exists an (i_0', t_0, x_0, y_0) *such that* $\sup_{i,t,x,y} \psi^{(i)}(t, x, y) = \psi^{(i_0')}(t_0, x_0, y_0)$*and* $\mathcal{J}_1 \cap \mathcal{J}_2(t_0, x_0) = \emptyset$*. Then, there exists an* i_0 *such that* $\psi^{(i_0)}(t_0, x_0, y_0) = \psi^{(i_0')}(t_0, x_0, y_0)$ *and*

$$v^{(i_0)}(t_0, y_0) < \mathcal{M}^{(i_0)} v(t_0, y_0).$$

Moreover, if in a neighborhood of (t_0, x_0, y_0) *there are some continuous functions* $h_0(t, x, y)$*,* $h(t, x)$ *and* $\hat{h}(t, y)$ *such that*

$$D^2 \varphi(t, x, y) \leq h_0(t, x, y) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} h(t, x) & 0 \\ 0 & \hat{h}(t, y) \end{pmatrix},$$

then there are $a, b \in \mathbb{R}$ *and* $X, Y \in \mathbb{S}_+^d$ *such that*

$$\begin{aligned}
a - b &= \varphi_t(t_0, x_0, y_0), \\
\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} &\leq 2h_0(t_0, x_0, y_0) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} h(t_0, x_0) & 0 \\ 0 & \hat{h}(t_0, y_0) \end{pmatrix},
\end{aligned}$$

$$-a+\inf_{\alpha\in\mathcal{A}_{i_0}}\left\{\frac{1}{2}\mathrm{Tr}\left[a^{\alpha}_{i_0}(t_0,x_0)X\right]+b^{\alpha}_{i_0}(t_0,x_0)D_x\varphi(t_0,x_0,y_0)+c^{\alpha}_{i_0}(t_0,x_0)u^{(i_0)}(t_0,x_0)+f^{\alpha}_{i_0}(t_0,x_0)\right\}\le 0, \tag{A.5}$$

$$-b+\inf_{\alpha\in\mathcal{A}_{i_0}}\left\{\frac{1}{2}\mathrm{Tr}\left[\hat{a}^{\alpha}_{i_0}(t_0,y_0)Y\right]+\hat{b}^{\alpha}_{i_0}(t_0,y_0)(-D_y\varphi(t_0,x_0,y_0))+\hat{c}^{\alpha}_{i_0}(t_0,y_0)v^{(i_0)}(t_0,y_0)+\hat{f}^{\alpha}_{i_0}(t_0,y_0)\right\}\ge 0. \tag{A.6}$$

Proof. The first part of the proof is similar to those of Lemma A.2 in [2], Lemma A.1 in [6]. The second part follows as a result of Theorem 2.2 in [10]. □

The following theorem on continuous dependence is used in Section 3.3.2 and in the regularity result, Theorem A.6 below. Intuitively speaking, continuous dependence result asserts that a slight change in the coefficients of (A.1) changes the solution only slightly.

Theorem A.3 (Continuous dependence)**.** *Let* **HJB-S** *hold. Suppose that* $u=(u^{(i)})_i$ *and* $v=(v^{(i)})_i$ *are a bounded upper semicontinuous subsolution of* (A.1) *and a bounded lower semicontinuous supersolution of* (A.2)*, respectively. Then, for any* $i=1,\cdots,N$*,*

$$u^{(i)}-v^{(i)}\le\mathcal{B}:=C\max_j\Bigg\{|(g-\hat{g})(\cdot,\cdot)|_\infty+T\sup_\alpha\left\{|f^{j,\alpha}-\hat{f}^{j,\alpha}|_\infty+(|u|_\infty\vee|v|_\infty)|c^{j,\alpha}-\hat{c}^{j,\alpha}|_\infty\right\}$$
$$+\sqrt{T}\sup_\alpha\left\{|\sigma^{j,\alpha}-\hat{\sigma}^{j,\alpha}|_\infty+|b^{j,\alpha}-\hat{b}^{j,\alpha}|_\infty\right\}\Bigg\}.$$

Proof. Let $\varphi(t,x,y)=e^{\lambda(T-t)}\frac{\theta}{2}|x-y|^2+e^{\lambda(T-t)}\frac{\varepsilon}{2+\gamma}(|x|^{2+\gamma}+|y|^{2+\gamma})$ and define

$$\mathcal{D}:=\sup_{t,i,x,y}\left\{u^{(i)}(t,x)-v^{(i)}(t,y)-\varphi(t,x,y)-\frac{\bar{\varepsilon}}{t}\right\},$$

where $\varepsilon,\bar{\varepsilon},\theta>0$ are arbitrary constants and constants λ and $\gamma>0$ will be determined later in the proof. We will show that $\mathcal{D}$ is bounded by a constant $B(\varepsilon,\bar{\varepsilon},\theta)$ which converges to the bound $\mathcal{B}$ mentioned in the theorem as $(\varepsilon,\bar{\varepsilon},\theta)\to(0,0,\infty)$. From the latter it would follow that

$$u^{(i)}(t,x)-v^{(i)}(t,x)\le\mathcal{D}+\frac{\bar{\varepsilon}}{t}+e^{\lambda(T-t)}\frac{2\varepsilon|x|^{2+\gamma}}{2+\gamma}\le B(\varepsilon,\bar{\varepsilon},\theta)+\frac{\bar{\varepsilon}}{t}+e^{\lambda(T-t)}\frac{2\varepsilon|x|^{2+\gamma}}{2+\gamma}.$$

Sending $\varepsilon,\bar{\varepsilon}\to 0$ and $\theta\to\infty$ one would then obtain

$$u^{(i)}(t,x)-v^{(i)}(t,x)\le\mathcal{B},\ \text{for } t>0.$$

Note that the above inequality is also valid for $t=0$ by considering $[-\delta,T]$ as the time interval by changing T to $T+\delta$.

Define

$$\psi^{(i)}(t,x,y)=u^{(i)}(t,x)-v^{(i)}(t,y)-\varphi(t,x,y)-\frac{\sigma(T-t)}{2T}-\frac{\bar{\varepsilon}}{t},$$

where $\sigma=\mathcal{D}-\sigma_T$ with $\sigma_T=\sup_{i,x,y}\left\{u^{(i)}(T,x)-v^{(i)}(T,y)-\varphi(T,x,y)-\frac{\bar{\varepsilon}}{T}\right\}^+$. Let

$$\bar{\mathcal{D}}:=\sup_{i,t,x,y}\psi^{(i)}(t,x,y). \tag{A.7}$$

Since $u^{(i)}$ and $v^{(i)}$ are bounded, we have $\bar{\mathcal{D}} < \infty$. On the other hand, by semicontinuity of $u^{(i)}$ and $v^{(i)}$, one can conclude that the supremum in the definition of $\bar{\mathcal{D}}$ is attained at some point (i_0, t_0, x_0, y_0). In other words, $\mathcal{J}_1 \neq \emptyset$ (see Lemma A.2 for the definition of $\mathcal{J}_1$).

If $\sigma \leq 0$, then $\mathcal{D} \leq \sigma_T$. Since

$$\begin{aligned}\sigma_T &\leq |g-\hat{g}|_\infty + \sup_{x,y}\{|g|_1|x-y| - \varphi(T,x,y)\} - \frac{\bar{\varepsilon}}{T} \\ &\leq |g-\hat{g}|_\infty + \sup_{x,y}\{|g|_1|x-y| - \frac{\theta}{2}|x-y|^2\} \leq |g-\hat{g}|_\infty + \frac{|g|_1}{2\theta},\end{aligned}$$

one can conclude that $\mathcal{D} \leq |g-\hat{g}|_\infty + \frac{|g|_1}{2\theta}$. Therefore, we may assume that $\sigma > 0$. From the definition of $\bar{\mathcal{D}}$, we have $t_0 > 0$. On the other hand, $\sigma > 0$ implies $t_0 < T$. Because if $t_0 = T$, then $\sigma_T \geq \bar{\mathcal{D}}$ which implies

$$\sigma_T \geq \mathcal{D} - \frac{\sigma}{2} \geq \sigma_T + \frac{\sigma}{2} > \sigma_T$$

which is a contradiction. So, we have $0 < t_0 < T$. We continue the proof by considering two different cases.

Case 1: $\mathcal{J}_1 \cap \mathcal{J}_2(t_0, x_0) \neq \emptyset$. The supremum in (A.7) is attained at some point (i_0, t_0, x_0, y_0) with $u^{(i)}(t_0, x_0) \leq g(t_0, x_0)$ and $v^{(i)}(t_0, y_0) \geq \hat{g}(t_0, y_0)$. Therefore,

$$\begin{aligned}\bar{\mathcal{D}} &\leq g(t_0,x_0) - \hat{g}(t_0,y_0) - \varphi(t_0,x_0,y_0) - \frac{\sigma\,(T-t_0)}{2T} - \frac{\bar{\varepsilon}}{t_0} \\ &\leq |g-\hat{g}|_\infty + |g|_1|x_0-y_0| - \frac{\theta}{2}|x_0-y_0|^2 \leq |g-\hat{g}|_\infty + \frac{|g|_1}{2\theta}.\end{aligned}$$

On the other hand, since $\mathcal{D} \leq \bar{\mathcal{D}} + \frac{\sigma}{2} \leq \bar{\mathcal{D}} + \frac{1}{2}(\mathcal{D} - \sigma_T) \leq \bar{\mathcal{D}} + \frac{1}{2}\mathcal{D}$, we have $\mathcal{D} \leq |g-\hat{g}|_\infty + \frac{|g|_1}{\theta}$.

Case 2: $\mathcal{J}_1 \cap \mathcal{J}_2(t_0, x_0) = \emptyset$. By Lemma A.2, there exists $a, b \in \mathbb{R}$ and $X, Y \in \mathbb{S}^d_+$ such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq e^{\lambda(T-t_0)}\theta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + e^{\lambda(T-t_0)}\frac{\varepsilon}{2+\gamma}(|x_0|^\gamma + |y_0|^\gamma)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{A.8}$$

$$a - b = -\lambda\varphi(t_0,x_0,y_0) - \frac{\sigma}{2T} - \frac{\bar{\varepsilon}}{t_0^2}, \tag{A.9}$$

and (A.5), (A.6) hold. Using Assumption **HJB-S** and (A.8), one can write

$$\begin{aligned}&\mathrm{Tr}\left[a^{\alpha,i_0}(t_0,x_0)X - \hat{a}^{\alpha,i_0}(t_0,y_0)Y\right] \\ &\leq 2e^{\lambda(T-t_0)}\theta\,\mathrm{Tr}\left[(\sigma^{\alpha,i_0}(t_0,x_0) - \hat{\sigma}^{\alpha,i_0}(t_0,y_0))^2\right] + \varepsilon e^{\lambda(T-t_0)}(|x_0|^\gamma + |y_0|^\gamma)(\mathrm{Tr}\left[a^{\alpha,i_0}(t_0,x_0)) + \hat{a}^{\alpha,i_0}(t_0,y_0)\right]) \\ &\leq C_1 e^{\lambda(T-t_0)}\left(\theta|x_0-y_0|^2 + \theta|\sigma^{\alpha,i_0} - \hat{\sigma}^{\alpha,i_0}|^2_\infty + \varepsilon(1+|x_0|+|y_0|)^{1+\gamma}\right),\end{aligned}$$

$$\begin{aligned}&e^{\lambda(T-t_0)}\left(b^{\alpha,i_0}(t_0,x_0)(\theta(x_0-y_0) + \varepsilon|x_0|^{1+\gamma}) + \hat{b}^{\alpha,i_0}(t_0,y_0)((\theta(y_0-x_0) + \varepsilon|y_0|^{1+\gamma})\right) \\ &\qquad \leq C_2 e^{\lambda(T-t_0)}\left(\theta|x_0-y_0|^2 + \theta|b^{\alpha,i_0} - \hat{b}^{\alpha,i_0}|_\infty|x_0-y_0| + \varepsilon(1+|x_0|+|y_0|)^{2+\gamma}\right),\end{aligned}$$

and

$$f^{\alpha,i_0}(t_0,x_0) - \hat{f}^{\alpha,i_0}(t_0,y_0) \leq C_4(|x_0-y_0| + |f^{\alpha,i_0} - \hat{f}^{\alpha,i_0}|_\infty).$$

Observe that since $0 < \sigma \leq \mathcal{D} \leq \bar{\mathcal{D}} + \frac{\sigma}{2}$, we have $0 < \bar{\mathcal{D}}$ which implies $u^{(i_0)}(t_0,x_0) - v^{(i_0)}(t_0,y_0) > 0$. Because $c^{\alpha,i} \geq 0$,

$$\hat{c}^{\alpha,i_0}(t_0,y_0)v^{i_0}(t_0,y_0) - c^{\alpha,i_0}(t_0,x_0)u^{i_0}(t_0,x_0) \leq C_3(|u^{(i)}|_\infty \vee |v^{(i)}|_\infty)(|\hat{c}^{\alpha,i_0} - \hat{c}^{\alpha,i_0}|_\infty + |x_0 - y_0|).$$

Subtracting (A.6) from (A.5) and using the estimates above, we obtain

$$\begin{aligned}\lambda\varphi(t_0,x_0,y_0) + \frac{\sigma}{2T} \leq C\Big(&\theta|x_0-y_0|^2 + \theta|\sigma^{\alpha,i_0} - \hat{\sigma}^{\alpha,i_0}|_\infty^2 \\ &+ \theta|b^{\alpha,i_0} - \hat{b}^{\alpha,i_0}|_\infty|x_0-y_0| + \varepsilon(1+|x_0|+|y_0|)^{2+\gamma} \\ &+ (|u^{(i)}|_\infty \vee |v^{(i)}|_\infty)(|\hat{c}^{\alpha,i_0} - \hat{c}^{\alpha,i_0}|_\infty + |x_0-y_0|) \\ &+ |f^{\alpha,i_0} - \hat{f}^{\alpha,i_0}|_\infty\Big),\end{aligned} \tag{A.10}$$

where $C = \max\{C_1, C_2, C_3, C_4\}$. On the other hand, since $\sigma_T \leq |g-\hat{g}|_\infty + \frac{|g|_1^2}{2\theta}$, one can conclude from (A.10), which gives an upper bound on σ, that

$$\begin{aligned}\mathcal{D} \;\leq\;& \sigma + \sigma_T \leq |g-\hat{g}|_\infty + \frac{|g|_1^2}{2\theta} - 2\lambda T\varphi(t_0,x_0,y_0) + CT\Big(\theta|x_0-y_0|^2 \\ &+\theta|\sigma^{\alpha,i_0} - \hat{\sigma}^{\alpha,i_0}|_\infty^2 + \theta|b^{\alpha,i_0} - \hat{b}^{\alpha,i_0}|_\infty|x_0-y_0| + \varepsilon(1+|x_0|+|y_0|)^{2+\gamma} \\ &+(|u^{(i)}|_\infty \vee |v^{(i)}|_\infty)(|\hat{c}^{\alpha,i_0} - \hat{c}^{\alpha,i_0}|_\infty + |x_0-y_0|) + |f^{\alpha,i_0} - \hat{f}^{\alpha,i_0}|_\infty\Big).\end{aligned}$$

Choosing λ large enough and maximizing with respect to $|x_0 - y_0|$

$$\begin{aligned}\mathcal{D} \;\leq\;& |g-\hat{g}|_\infty + \frac{|g|_1^2}{2\theta} + CT\Big(\theta|b^{\alpha,i_0} - \hat{b}^{\alpha,i_0}|_\infty^2 + \theta|\sigma^{\alpha,i_0} - \hat{\sigma}^{\alpha,i_0}|_\infty^2 \\ &+(|u^{(i)}|_\infty \vee |v^{(i)}|_\infty)|\hat{c}^{\alpha,i_0} - \hat{c}^{\alpha,i_0}|_\infty + |f^{\alpha,i_0} - \hat{f}^{\alpha,i_0}|_\infty\Big).\end{aligned}$$

Now maximizing with respect to θ yields the desired result. □

The following result in a straightforward consequence of Theorem A.3, and will be used to establish the existence and the regularity of the solution to (A.1).

Corollary A.4. *Assume that* **HJB-S** *holds. Suppose that* $u = (u^{(i)})_i$ *and* $v = (v^{(i)})_i$ *are respectively a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution of* (A.1)*. Then, for any* $i = 1, \cdots, N$, $u^{(i)} \leq v^{(i)}$.

Theorem A.5 (Existence)**.** *Assume that* **HJB-S** *holds. Then there exists a unique continuous viscosity solution in the class of bounded functions to* (A.1)*.*

Proof. We follow Perron's method (see e.g. Section 4 in [10]). Observe that by Assumption **HJB-S**, $\underline{u} = -K$ and $\overline{v} = K$ are respectively sub and supersolutions of (A.1) for a suitable choice of positive constant K. Define $v^{(i)}(t,x) := \sup\{u^{(i)}(t,x)\ ;\ u \text{ is a subsolution to (A.1) }\}$ and

$$v^{(i)*}(t,x) := \lim_{\delta\to 0}\sup \sup\{v(s,y) : |x-y| + |s-t| \leq \delta, s \in [0,T]\},$$

and

$$v_*^{(i)}(t,x) := \liminf_{\delta\to 0} \inf\{v(s,y) : |x-y| + |s-t| \le \delta, s \in [0,T]\}.$$

It is straight forward that $-K \le v_*^{(i)} \le v^{(i)*} \le K$. We want to show that $(v^{(i)*})_{i=1}^M$ and $(v_*^{(i)})_{i=1}^M$ are respectively a sub and a supersolution to (A.1) which by comparison, Corollary A.4, yields the desired result.

Step 1: Subsolution property of $v^{(i)*}$. We start by showing that $(U, \cdots, U)$ with

$$U(t,x) := a_\varepsilon(T-t) + g(T,z) + |g|_1 \left(T - t + |x-z|^2 + \varepsilon\right)^{\frac{1}{2}}$$

is a supersolution to(A.1) for a suitable positive constant a_ε. Observe that since

$$U(t,x) - g(t,x) \ge g(T,z) - g(t,x) + |g|_1 \left(T - t + |x-z|^2 + \varepsilon\right)^{\frac{1}{2}} \ge 0,$$

we have that $U(t,x) \ge g(t,x)$, and in particular $U(T,x) \ge g(T,x)$. On the other hand, by simple calculations, one can show that, for an appropriate choice of a_ε, we have $-U_t - \inf_{\alpha\in\mathcal{A}^i} \mathcal{L}^{i,\alpha} U \ge 0$.

Therefore, by comparison, Corollary A.4, for any subsolution u, $u \le U$ which implies $v^{(i)*} \le U$; specially $v^{(i)*}(T,x) \le U(T,x)$. Sending $\varepsilon \to 0$ and setting $x = z$, $v^{(i)*}(T,x) \le g(T,x)$.

Now, for fixed i, we suppose $t < T$ and φ is a test function such that

$$0 = \max_{[0,T]\times\mathbb{R}^d} \{v^{(i)*} - \varphi\} = (v^{(i)*} - \varphi)(t,x).$$

It follows from the definition of $v^{(i)*}$ that there exists a sequence $\{(u_n, t_n, x_n)\}_n$ with $t_n < T$ such that u_n is a subsolution to (A.1), $(t_n, x_n) \to (t,x)$, $u_n^{(i)}(t_n,x_n) \to v^{(i)*}(t,x)$, and (t_n,x_n) is the global strict maximum of $u_n^{(i)} - \varphi$. Let $\delta_n := \max_{[0,T]\times\mathbb{R}^d}\{u_n^{(i)} - \varphi\}$. By the subsolution property of u_n, we have

$$\min\left\{\max\left\{-\varphi_t - \inf_{\alpha\in\mathcal{A}^i} \mathcal{L}^{i,\alpha}(\varphi + \delta_n), u_n^{(i)} - \mathcal{M}^{(i)} u_n\right\}, \varphi + \delta_n - g\right\}(t_n,x_n) \le 0.$$

Because $\mathcal{M}^{(i)} u_n \le \mathcal{M}^{(i)} v^*$, by sending $n \to \infty$,

$$\min\left\{\max\left\{-\varphi_t - \inf_{\alpha\in\mathcal{A}^i} \mathcal{L}^{i,\alpha}\varphi, v^{(i)*} - \mathcal{M}^{(i)} v^*\right\}, v^{(i)*} - g\right\}(t,x) \le 0.$$

Step 2: Supersolution property of $v_*^{(i)}$. Since $(g, \cdots, g)$ is a subsolution to (A.1), $v_*^{(i)}(t,x) \ge g(t,x)$. In particular, $v_*^{(i)}(T,x) \ge g(T,x)$. Therefore, we only need to show that

$$\max\left\{-(v_*^{(i)})_t - \inf_{\alpha\in\mathcal{A}^i} \mathcal{L}^{i,\alpha} v_*^{(i)}, v_*^{(i)} - \mathcal{M}^{(i)} v_*\right\} \ge 0, \tag{A.11}$$

on $[0,T) \times \mathbb{R}^d$ in the viscosity sense. We will prove (A.11) by a contradiction argument. Assume that there are a test function φ and (i,t,x) with $t < T$ such that (t,x) is the global strict minimum of $v_*^{(i)} - \varphi$ and $\max\left\{-\varphi_t - \inf_{\alpha\in\mathcal{A}^i} \mathcal{L}^{i,\alpha}\varphi, \varphi - \mathcal{M}^{(i)} v_*\right\}(t,x) < 0$. Then, by continuity of φ and the equation and lower semicontinuity of v_*, one can find $\varepsilon > 0$ and $\delta > 0$ small enough, such that for $|x-y| + |s-t| < \delta$ we have that $\varphi + \varepsilon < v_*^{(i)}$ and that

$$\max\left\{-(\varphi+\varepsilon)_t - \inf_{\alpha\in\mathcal{A}^i} \mathcal{L}^{i,\alpha}(\varphi+\varepsilon), (\varphi+\varepsilon) - \mathcal{M}^{(i)} v_*\right\}(s,y) < 0. \tag{A.12}$$

Define

$$w^{(j)}(s,y) := \begin{cases} \max\{\varphi+\varepsilon, v^{(j)*}\}(s,y), & j=i \ \text{ and } \ |x-y|+|s-t|<\delta; \\ v^{(j)*}(s,y), & \text{otherwise.} \end{cases}$$

Since v^* is a subsolution to (A.1) and by (A.12), one can show that w is a subsolution to (A.1). By the definition of $v_*^{(i)}$, we must have $v_*^{(i)} \geq w^{(i)}$, which contradicts with the fact that $w^{(i)}(t,x) = \varphi(t,x)+\varepsilon < v_*^{(i)}(t,x)$ for $|x-y|+|s-t|<\delta$. □

Theorem A.6 (Regularity)**.** *Assume that* **HJB-S** *holds. Let* $(u^{(i)})_{i=1}^M$ *be the solution to* (A.1)*. Then,* $(u^{(i)})_{i=1}^M$ *is Lipschitz continuous with respect to* x *and* $\frac{1}{2}$*-Hölder continuous with respect to* t *on* $\mathbb{R}^d \times [0,T]$*.*

Proof. Lipschitz continuity with respect to x**:** For fixed $y \in \mathbb{R}^d$, $v^{(i)}(x) = u^{(i)}(t,x+y)$ is the solution of a switching system obtained from (A.1) by replacing $\mathcal{L}^{i,\alpha}$ with

$$\mathcal{L}^{i,\alpha,y}\varphi(x) \quad := \quad \frac{1}{2}\mathrm{Tr}\left[a_i^\alpha(t,x+y)D^2\varphi\right] + b_i^\alpha(t,x+y)D\varphi + c_i^\alpha(t,x+y)\varphi + f_i^\alpha(t,x+y),$$

with the terminal condition given by $v^{(i)}(T,x) = g(T,x+y)$. By Theorem A.3, there is a positive constant C such that

$$|u^{(i)}(t,x) - u^{(i)}(t,x+y)|_\infty = |u^{(i)}(t,x) - v^{(i)}(t,x)|_\infty \quad \leq \quad C|y|.$$

$\frac{1}{2}$**-Hölder continuity with respect to** t**:** For $t<s$, define $\bar{u} = (\bar{u}^{(i)})_{i=1}^M$ to be the solution to

$$\max\left\{-\bar{u}_t^{(i)} - F_i(\cdot, \bar{u}^{(i)}, D\bar{u}^{(i)}, D^2\bar{u}^{(i)}), \bar{u}^{(i)} - \mathcal{M}^{(i)}\bar{u}\right\} = 0, \quad \text{for } i=1,\cdots,M;$$
$$\bar{u}^{(i)}(s,\cdot) = u^{(i)}(s,\cdot).$$

Since $\bar{u}$ is a subsolution of (A.1) on $[0,s]\times\mathbb{R}^d$ with terminal condition $u^{(i)}(s,\cdot)$, by comparison result, Corollary A.4, we have $\bar{u}^{(i)} \leq u^{(i)}$. Therefore, $u^{(i)}(t,x) - u^{(i)}(s,x) \geq \bar{u}^{(i)}(t,x) - \bar{u}^{(i)}(s,x)$. By Theorem A.1 in [3], $\bar{u}^{(i)}$ is $\frac{1}{2}$-Hölder continuous in t which provides

$$u^{(i)}(t,x) - u^{(i)}(s,x) \geq -C\sqrt{s-t}.$$

Now, for fixed $y \in \mathbb{R}^d$, define

$$\psi^{(i)}(t,x) := \frac{\lambda L}{2}e^{A(s-t)}\left(|x-y|^2 + B(s-t)\right) + \frac{L}{\lambda} + B(s-t) + g(s,y),$$

where A, B and λ are positive constants which will be given later and L is the same as in Assumption **HJB-S**. We will show that for an appropriate choice of A and B, $(\psi^{(i)})_{i=1}^M$ is a supersolution of (A.1) with terminal condition $g(s,x)$. Then, comparison, Corollary A.4, would then imply that $u^{(i)} \leq \psi^{(i)}$. Therefore,

$$u^{(i)}(t,y) - u^{(i)}(s,y) \quad \leq \quad \psi^{(i)}(t,y) - g(s,y) \leq \frac{\lambda L}{2}e^{A(s-t)}B(s-t) + \frac{L}{\lambda} + B(s-t).$$

By setting $\lambda = \frac{1}{\sqrt{s-t}}$, we have $u^{(i)}(t,y) - u^{(i)}(s,y) \leq C\sqrt{s-t}$, where C is a positive constant.

Therefore, it remains to show that for A and B large enough, we have

$$\min\left\{\max\left\{-\psi_t^{(i)} - \inf_{\alpha\in\mathcal{A}^i}\mathcal{L}^{i,\alpha}\psi^{(i)}, \psi^{(i)} - \mathcal{M}^{(i)}\psi^{(i)}\right\}, \psi^{(i)} - g\right\} \geq 0,$$

on $[0,s]\times\mathbb{R}^d$. Since $\psi^{(i)}-\mathcal{M}^{(i)}\psi^{(i)}<0$, one needs to show that

$$-\psi^{(i)}_t-\inf_{\alpha\in\mathcal{A}^i}\mathcal{L}^{i,\alpha}\psi^{(i)}\geq 0\quad\text{and}\quad\psi^{(i)}-g\geq 0.$$

Observe that if $B\geq 1$, by the regularity assumption on g, we have

$$\psi^{(i)}(t,x)-g(t,x)\quad\geq\quad\frac{L}{2}\left(\lambda|x-y|^2+\lambda(s-t)+\frac{2}{\lambda}\right)+g(s,y)-g(t,x)\geq 0.$$

On the other hand,

$$\begin{aligned}-\psi^{(i)}_t-\inf_{\alpha\in\mathcal{A}^i}\mathcal{L}^{i,\alpha}\psi^{(i)}=\sup_{\alpha\in\mathcal{A}^i}\Bigg\{&\frac{L\lambda}{2}e^{A(s-t)}\Bigg(A|x-y|^2+AB(s-t)+B\\&-\frac{1}{2}\mathrm{Tr}\left[a^{\alpha,i}\right]-b^{\alpha,i}\cdot(x-y)\Bigg)+B-c^{\alpha,i}\psi^{(i)}-f^{\alpha,i}\Bigg\}\\&\geq\frac{L\lambda}{2}e^{A(s-t)}\left(A|x-y|^2-L|x-y|+LB-\frac{L}{2}\right)+B-CL.\end{aligned}$$

By choosing A and B large enough, the right hand side in the above inequality is positive which completes the argument. □

References

[1] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4(3):271–283, 1991.

[2] Guy Barles and Espen R. Jakobsen. Error bounds for monotone approximation schemes for Hamilton-Jacobi-Bellman equations. *SIAM J. Numer. Anal.*, 43(2):540–558 (electronic), 2005.

[3] Guy Barles and Espen R. Jakobsen. Error bounds for monotone approximation schemes for parabolic Hamilton-Jacobi-Bellman equations. *Math. Comp.*, 76(260):1861–1893 (electronic), 2007.

[4] Guy Barles and Espen Robstad Jakobsen. On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations. *M2AN Math. Model. Numer. Anal.*, 36(1):33–54, 2002.

[5] E. Bayraktar and Y.-J. Huang. On the Multi-Dimensional Controller and Stopper Games. *ArXiv e-prints*, September 2010.

[6] J. Frédéric Bonnans, Stefania Maroso, and Housnaa Zidani. Error estimates for stochastic differential games: the adverse stopping case. *IMA J. Numer. Anal.*, 26(1):188–212, 2006.

[7] Bruno Bouchard and Jean-François Chassagneux. Discrete-time approximation for continuously and discretely reflected BSDEs. *Stochastic Process. Appl.*, 118(12):2269–2293, 2008.

[8] Bruno Bouchard and Nizar Touzi. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.*, 111(2):175–206, 2004.

[9] Bouchard Bruno and Xavier Warin. Monte-Carlo valorisation of American options: facts and new algorithms to improve existing methods. *To appear in Numerical Methods in Finance , Springer Proceedings in Mathematics, ed. R. Carmona, P. Del Moral, P. Hu and N. Oudjane.*

[10] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[11] A. Fahim. A probabilistic numerical method for fully non–linear elliptic and parabolic pdes in general domains. Preprint, Sep 2011.

[12] A. Fahim, N. Touzi, and X. Warin. A probabilistic numerical method for fully non–linear parabolic pdes. *Annals of Applied Probability*, 21(4):1322–1364, 2011.

[13] Ioannis Karatzas and S. G. Kou. Hedging American contingent claims with constrained portfolios. *Finance Stoch.*, 2(3):215–258, 1998.

[14] N. V. Krylov. On the rate of convergence of finite-difference approximations for Bellman's equations. *Algebra i Analiz*, 9(3):245–256, 1997.

[15] Nicolai V. Krylov. The rate of convergence of finite-difference approximations for Bellman equations with Lipschitz coefficients. *Appl. Math. Optim.*, 52(3):365–399, 2005.

[16] F.A. Longstaff and E.S. Schwartz. Valuing American options by simulation: a simple least-squares approach. *Review of Financial Studies*, 14(1):113–147, 2001.

[17] Jin Ma and Jianfeng Zhang. Representations and regularities for solutions to BSDEs with reflections. *Stochastic Process. Appl.*, 115(4):539–569, 2005.

[18] Adam M. Oberman. Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problems. *SIAM J. Numer. Anal.*, 44(2):879–895 (electronic), 2006.

[19] Adam M. Oberman and Thaleia Zariphopoulou. Pricing early exercise contracts in incomplete markets. *Computational Management Science*, 1(1):75–107, 2003.

[20] Jianfeng Zhang. A numerical scheme for BSDEs. *Ann. Appl. Probab.*, 14(1):459–488, 2004.

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